AN INTERESTING EXAMPLE FOR SPECTRAL INVARIANTS
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Abstract. In [HL99], the heat operator of a Bismut superconnection for a family of generalized Dirac operators is defined along the leaves of a foliation with Hausdorff groupoid. The Novikov-Shubin invariants of the Dirac operators were assumed greater than three times the codimension of the foliation. It was then showed that the associated heat operator converges to the Chern character of the index bundle of the operator. In [BH08], we improved this result by reducing the requirement on the Novikov-Shubin invariants to one half of the codimension. In this paper, we construct examples which show that this is the best possible result.

1. Introduction

In [HL99], Connor Lazarov and the second author gave general conditions which guarantee that the Bismut superconnection formalism extends in full generality to families of non-compact manifolds. This allowed us to prove a families index theorem for generalized Dirac operators defined along such families. Special cases give the Atiyah-Singer families index theorem, the Atiyah L^2 index theorem [A76], and the Connes foliation cohomology index theorem [C86]. In addition, we got a new theorem for fiber bundles which is a combination of the first two above, namely an L^2 families index theorem. Note that Connes and Skandalis have proven a families index theorem for families of elliptic operators defined along the leaves of a foliation. See [CS84] and [C87]. One of the conditions we required is that the Novikov-Shubin invariants, [NS86a, NS86b, HL99], of the Dirac operators were greater than three times the codimension of the foliation.

In [BH08], we extended these results to generalized Dirac operators D along the leaves of a Riemannian foliation. We assumed that the projection onto the kernel of D is transversely smooth, and that the spectral projections of D^2 for the intervals $(0,\epsilon)$ are transversely smooth, for ϵ sufficiently small. We defined the Chern-Connes character of D in the Haefliger cohomology of the foliation. We then showed that the pairing of this Chern-Connes character with a given Haefliger $2k-$current is the same as that of the Haefliger Chern character of the index bundle of D, whenever the Novikov-Shubin invariants of D are greater than k. In particular, if the Novikov-Shubin invariants are greater than half the codimension of F, they are always the same. We conjectured that this theorem is still true provided only that the Novikov-Shubin invariants are positive. In this paper, we show that this is false. It is an interesting question as to what additional conditions need to be imposed for the conjecture to be true.

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2. A bit of background

We assume that the reader is familiar with the paper [BH08], in particular with the concepts of that paper, including: Haefliger cohomology; generalized Dirac operators; the various Chern-Connes characters used there; the Novikov-Shibin invariants; and transverse smoothness of leafwise operators. Suppose that $\widehat{D}$ is a generalized Dirac operator defined along the leaves of a Riemannian foliation F of codimension n. Denote by D the induced leafwise operator for the foliation F_s on the holonomy (or homotopy) groupoid of

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F. Denote by P_0 the leafwise projection onto the kernel of D, and by P_ϵ the leafwise spectral projection for D^2 for the interval $(0, \epsilon)$.

Theorem 2.1 (Theorem 4.2 of [BH08]). *Assume that P_0, and (for ϵ sufficiently small) P_ϵ are transversely smooth. For a fixed integer k with $0 \leq k \leq n/2$, assume that the Novikov-Shubin invariants of D are greater than k. Then the k^{th} component of the Chern character of the K-theory index of D equals the k^{th} component of the Chern character of the index bundle of D, that is, in the Haefliger cohomology $\mathrm{H}_c^{2k}(M/F)$ of F,*

$$\mathrm{ch}_a^k(\mathrm{Ind}_a(D^+)) = \mathrm{ch}_a^k([P_0]).$$

Corollary 2.2 (Theorem 4.1 of [BH08]). *If the Novikov-Shubin invariants of D are greater than $n/2$, then*

$$\mathrm{ch}_a(\mathrm{Ind}_a(D^+)) = \mathrm{ch}_a([P_0]).$$

3. The examples

We now show that Corollary 2.2 is the best possible.

Consider the product of two n-tori $\mathbb{T}^n \times \mathbb{T}^n$, and a foliation F on it which is the product of constant irrational slope slope foliations on the individual $\mathbb{T} \times \mathbb{T}$. We assume that the slopes are not rationally related. Then F is a Riemannian foliation, and its holonomy and homotopy groupoids are just the product $\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{R}^n$, with the foliation F_s given by the $\mathbb{R}^n$ factors.

Denote by $\widehat{E}_n \to \mathbb{T}^n \times \mathbb{T}^n$ the Hermitian bundle which is given as follows. Let $\xi \in \mathbb{Z}^n = \pi_1(\mathbb{T}^n)$, and $(x, w, z) \in \mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C}$, and define

$$\xi \cdot (x, w, z) = (x + \xi, w, (\xi w) \cdot z),$$

where

$$(\xi w) \cdot z = (\exp(2\pi i \xi_1 w_1) z_1, \ldots, \exp(2\pi i \xi_n w_n) z_n).$$

Then

$$\widehat{E}_n = (\mathbb{R}^n \times \mathbb{T}^n \times \mathbb{C})/\mathbb{Z}^n.$$

Note that $\widehat{E}_n = E_1 \otimes \cdots \otimes E_n$, where E_j is the pull back of $\widehat{E}_1$ by the projection $\pi_j : \mathbb{T}^n \times \mathbb{T}^n \to \mathbb{T} \times \mathbb{T}$ onto the j-th coordinates. In [BH11], the proof of Theorem 11.3, we showed that $\mathrm{ch}(E_j) = 1 + \beta_j$, where β_j is the pull back under π_j of the natural generator of $H^2(\mathbb{T}^2; \mathbb{Z})$. Then the Chern character

$$\mathrm{ch}(\widehat{E}_n) \;=\; \prod_{j=1}^{n} \mathrm{ch}(E_j) \;=\; \prod_{j=1}^{n} 1 + \beta_j,$$

in particular, it is as non-trivial as it can be.

Next, note that $\widehat{E}_n$ restricted to the leaves of F is a flat bundle. This is because the leaves are products of $\mathbb{R}$s, where each $\mathbb{R}$ is contained in a leaf of the foliation of the j-th $\mathbb{T}^1 \times \mathbb{T}^1$, and the fact that the curvature of the pull back is the pull back of the curvature. But the connection we use is the product of the pull backs of the connection on $\mathbb{T}^1 \times \mathbb{T}^1$ by the π_j. Since the curvature of the pull back is the pull back of the curvature, the curvature of our connection on a leaf is the product of the pull backs of the curvatures on one dimensional manifolds, so this curvature is identically zero.

Denote by D_n the leafwise signature operator twisted by the leafwise flat bundle $\widehat{E}_n$, which is a generalized Dirac operator. This is a leafwise operator on the holonomy (homotopy) groupoid for the foliation F_s. But the leaves of F_s are all $\mathbb{R}^n$s, and the pull back of $\widehat{E}_n$ to the holonomy (homotopy) groupoid is just the trivial bundle $\mathbb{T}^n \times \mathbb{T}^n \times \mathbb{R}^n \times \mathbb{C}$. So D_n is just the the complexified leafswise signature operator. As there are no non-zero L^2 harmonic forms on $\mathbb{R}^n$, the projection onto the leafwise harmonics is the zero map, which is transversely smooth, and the Chern character of the index bundle $\mathrm{ch}_a([P_0])$ of D_n is zero. Finally, note that each point $x \in \mathbb{T}^n \times \mathbb{T}^n$ has a neighborhood U_x so that the structure over U_x is $U_x \times \mathbb{R}^n \times \mathbb{C}$ where the flat structure on $\mathbb{C}$ is the obvious one. It follows immediately that the spectral projections $P_{(0,\epsilon)}$ of D_n for the interval $(0, \epsilon)$ are constant in transverse directions so are tranversely smooth. Thus this example satisfies all the conditions of Corollary 2.2, except one. Namely, the Novikov-Shubin invariants of the signature operator

on $\mathbb{R}^n$ are exactly $n/2$, i.e. the Novikov-Shubin invariants of D_n are not greater than half the codimension of the foliation.

Recall, [HL99], Corollary 4, that the Haefliger Chern-Connes character of any leafwise Dirac operator with coefficients in a leafwise flat bundle E is given (up to a constant) by the integral over the fiber of the foliation of the characteristic class $\widehat{A}(TF)\,\mathrm{ch}(E)$, where $\widehat{A}(TF)$ is the $\widehat{A}$ genus of the tangent bundle of the foliation F. As TF is a trivial bundle, $\widehat{A}(TF)=1$, and the Haefliger Chern-Connes character of D_n is given by

$$\mathrm{ch}_a(D_n) \;=\; \int_F \mathrm{ch}(\widehat{E}_n).$$

By Hector et al, [EH86, EHS85], the Haefliger cohomology of F is the same as the basic cohomology , that is the cohomology of the transverse forms which are invariant under the holonomy. It is not hard to see that this isomorphic to $H^*(\mathbb{T}^n;\mathbb{R})$, and we can easily identify the class $\displaystyle\int_F \mathrm{ch}(\widehat{E}_n)$ under this isomorphism. In particular it is just $\prod_{j=1}^n \alpha_j$, where α_j is the pull back of the natural generator of $H^1(\mathbb{T}^1;\mathbb{Z})$ under the projection $\mathbb{T}^n \to \mathbb{T}^1$ onto the j th coordinate. Thus $\mathrm{ch}_a(D_n)$ is non-zero, so $\mathrm{ch}_a(\mathrm{Ind}_a(D^+)) \neq \mathrm{ch}_a([P_0])$ for this example. Note further that $\mathrm{ch}_a(D_n)$ is non-zero only in the top dimension, so for $k < n/2$, $\mathrm{ch}_a^k(D_n) = \mathrm{ch}_a^k([P_0])$, which they must be by Theorem 2.1.

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